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2007 J. Phys. A: Math. Theor. 40 1991

(http://iopscience.iop.org/1751-8121/40/9/006)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 1991–2011 doi:10.1088/1751-8113/40/9/006

Chaotic properties between the nonintegrable discrete nonlinear Schrödinger equation and a nonintegrable discrete Heisenberg model

Qing Ding

Institute of Mathematics and Key Laboratory of Mathematics for Nonlinear Sciences, Fudan University, Shanghai 200433, People's Republic of China

Received 9 October 2006, in final form 12 January 2007
Published 14 February 2007
Online at stacks.iop.org/JPhysA/40/1991

Abstract

We prove that the integrable–nonintegrable discrete nonlinear Schrödinger equation (AL-DNLS) introduced by Cai, Bishop and Gronbech-Jensen (*Phys. Rev. Lett.* **72** 591(1994)) is the discrete gauge equivalent to an integrable–nonintegrable discrete Heisenberg model from the geometric point of view. Then we study whether the transmission and bifurcation properties of the AL-DNLS equation are preserved under the action of discrete gauge transformations. Our results reveal that the transmission property of the AL-DNLS equation is completely preserved and the bifurcation property is conditionally preserved to those of the integrable–nonintegrable discrete Heisenberg model.

PACS numbers: 02.40.Ky, 05.45.Mt, 07.55.Db

1. Introduction

The nonlinear Schrödinger equation (NLSE): $\mathrm{i}\dot{q} + q'' + \kappa|q|^2q = 0$ (with $\kappa \neq 0$), where we use dot for the time derivative and the prime for the space derivative, is a prototypical integrable partial differential equation in mathematics which models a wide range of physical phenomena, such as nonlinear optical pulse propagation, hydrodynamics, biophysics and so on (see, for example, [1] for a list of the physical motivations of NLSE). Since most work in nonlinear wave propagation involves at some extent a numerical study of the problem, the issue of the discretization of the NLSE was addressed early in [2]. Among a large number of possible discretizations of the NLSE, Ablowitz and Ladik noticed that there is one which is also integrable [3]. It was shown that the integrable Ablowitz–Ladik (AL) equation has solutions which are essential discretization of classic soliton solutions to the NLSE [3]. Another discrete version of the NLSE was studied in [4–10] and references therein. The latter, usually referred to as the discrete nonlinear Schrödinger equation (DNLS) or discrete self-trapping equation, has quite a number of physical properties, but is in fact not integrable [4]. The motivations for studying the two discrete versions of the NLSE, i.e. the AL equation and the DNLS, are quite different: the AL equation, on the one hand, has very nice interesting mathematical properties, but not very clear physical significance; the introduction of the DNLS, on the other hand, is primarily considered physically (see, for example, [6, 8]).

An equation is introduced by Cai, Bishop and Gronbech-Jensen [7] that interpolates between the AL and the DNLS equation while containing these two equations as its limits. Its fundamental merit is that it allows us to study the interplay of the integrable and nonintegrable NLS-type terms in discrete lattice. Some detailed studies of the stationary version of the DNLS equation are displayed in [7–9]. On the other hand, the geometric exploitations of (discrete) gauge equivalence between integrable equations [11–14] or nonintegrable equations [15] play an important role in understanding the dynamics of those equations. Therefore, it is very interesting and important to have such a geometric study for the integrable–nonintegrable AL-DNLS equation, and, furthermore, to see whether quantum chaotic properties of the AL-DNLS equation are delivered or preserved under the action of discrete gauge transformations.

The paper is organized as follows. In section 2, we introduce the geometric concepts of discrete connection and associated discrete curvature. By using this geometric terminology, we prove that the AL-DNLS equation is discrete gauge equivalent to an integrable–nonintegrable discrete Heisenberg model. In section 3, we study whether the transmission and bifurcation properties of the AL-DNLS equation, which reflect quantum chaotic dynamics of the AL-DNLS equation, are delivered to the integrable–nonintegrable discrete Heisenberg equation under action of discrete gauge transformations and in section 4 we give conclusion and remarks. We set an appendix at the end of the paper to give detailed proofs of some identities which are important but not proved explicitly in the context of the paper.

2. Gauge equivalent structure of the AL-DNLS

The equation introduced by Cai, Bishop and Gronbech-Jensen [7] is as follows:

$$\mathrm{i}\dot{q}_n + (q_{n+1} + q_{n-1} - 2q_n) + \mu|q_n|^2(q_{n+1} + q_{n-1}) - \gamma|q_n|^2 q_n = 0, \tag{1}$$

where q_n is a complex amplitude, μ and γ are real constants called AL-nonlinearity and DNLS-nonlinearity strength, respectively. This equation interpolates two well-studied discretizations of the nonlinear Schrödinger equation, namely the AL and DNLS equations obtained by setting $\gamma = 0$ (with $\mu \neq 0$) and $\mu = 0$ (with $\gamma \neq 0$), respectively.

For the AL equation, we may normalize it by a scaling into the following form according to $\mu > 0$ (focusing) and $\mu < 0$ (de-focusing), respectively:

$$\mathrm{i}\dot{q}_n + (q_{n+1} + q_{n-1} - 2q_n) \pm |q_n|^2(q_{n+1} + q_{n-1}) = 0. \tag{2}$$

Lax pairs of equations (2) are ([13])

$$\phi_{n+1} = L_n\phi_n, \qquad \dot{\phi}_n = M_n\phi_n \tag{3}$$

in which

$$\begin{aligned} L_n &= \begin{pmatrix} z & \bar{q}_n z^{-1} \\ \mp q_n z & z^{-1} \end{pmatrix} \\ M_n &= \mathrm{i}\begin{pmatrix} 1 - z^2 + z - z^{-1} \mp \bar{q}_n q_{n-1} & -\bar{q}_n + \bar{q}_{n-1} z^{-2} \\ \mp q_n \pm q_{n-1} z^2 & -1 + z^{-2} + z - z^{-1} \pm q_n \bar{q}_{n-1} \end{pmatrix} \end{aligned} \tag{4}$$

where z is a spectral parameter and the overbar denotes complex conjugate. For the above Lax pairs (3), as usual (see, for example, [3, 13]), its continuous limit reads

$$\phi' = L\phi, \qquad \dot{\phi} = M\phi \tag{5}$$

with

$$L = \lambda\sigma_3 + U, \qquad M = -\mathrm{i}2\lambda^2\sigma_3 - 2\mathrm{i}\lambda U + \mathrm{i}(U^2 + U')\sigma_3,$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $U = \begin{pmatrix} 0 & \bar{q} \\ \mp q & 0 \end{pmatrix}$. It is a direct verification that the integrability condition of (5) yields just the NLSE: $\mathrm{i}\dot{q} + q'' \pm 2|q|^2 q = 0$.

It is proved in [12] (resp. [13]) that, as integrable discrete equations, the AL equation (2) of focusing (resp. de-focusing) type is gauge equivalent to the discrete Heisenberg spin model of $SU(2)$ type (resp. $SU(1,1)$ type):

$$\dot{S}_n = \mathrm{i}\frac{[S_{n+1}, S_n]}{1 + \mathbf{S}_{n+1}\cdot\mathbf{S}_n} - \mathrm{i}\frac{[S_n, S_{n-1}]}{1 + \mathbf{S}_n\cdot\mathbf{S}_{n-1}}, \qquad S_n = \begin{pmatrix} s_n^1 & s_n^2 - \mathrm{i}s_n^3 \\ s_n^2 + \mathrm{i}s_n^3 & -s_n^1 \end{pmatrix} \tag{6}$$

where $\mathbf{S}_n = (s_n^1, s_n^2, s_n^3) \in \mathbf{S}^2 \hookrightarrow \mathbf{R}^3$, i.e., $|\mathbf{S}_n|^2 = (s_n^1)^2 + (s_n^2)^2 + (s_n^3)^2 = 1$ and $\mathbf{S}_{n+1}\cdot\mathbf{S}_n$ is the inner product of the two vectors in R^3 (resp.

$$\dot{S}_n = -\frac{[S_{n+1}, S_n]}{1 - \mathbf{S}_{n+1}\cdot\mathbf{S}_n} + \frac{[S_n, S_{n-1}]}{1 - \mathbf{S}_n\cdot\mathbf{S}_{n-1}}, \qquad S_n = \begin{pmatrix} s_n^1 & \mathrm{i}(-s_n^2 + s_n^3) \\ \mathrm{i}(s_n^2 + s_n^3) & -s_n^1 \end{pmatrix} \tag{7}$$

where $\mathbf{S}_n = (s_n^1, s_n^2, s_n^3) \in \mathbf{H}^2 \hookrightarrow \mathbf{R}^{2+1}$, i.e., $|\mathbf{S}_n|^2 = (s_n^1)^2 + (s_n^2)^2 - (s_n^3)^2 = -1$ with $s_n^3 > 0$, and $\mathbf{S}_{n+1}\cdot\mathbf{S}_n$ in this case denotes the pseudo inner product of the three-dimensional Minkowski space $\mathbf{R}^{2+1}$ with metric signature $(+, +, -)$). The corresponding Lax pairs of equations (6) and (7) and some of their consequences are also presented in [12–14].

The geometric concept of gauge equivalence between integrable equations has been generalized to nonintegrable case in [15], where the gauge equivalent structure of (1+1)-dimensional anisotropic Landau–Lifshitz equation, regarded as an equation with prescribed non-zero $SU(2)$-curvature representation, is displayed. Following the idea in [15], we need to express equation (1) geometrically as a discrete equation with 'prescribed curvature representation'. So we must introduce some basic geometric concepts in the discrete case. Suppose that $\{L_n\}$ and $\{M_n\}$ are g-valued sequences depending on a time variable t and possibly a parameter z, where g is the Lie algebra of a given Lie group G. We define a discrete connection $\{A_n\}$ by

$$A_n = (L_n, M_n), \tag{8}$$

and its corresponding discrete curvature $\{F_n^A\}$ by

$$F_n^A = \dot{L}_n - M_{n+1}L_n + L_n M_n. \tag{9}$$

Now, for the AL equation (2), because of Lax pairs (3), its discrete connection $\{A_n\}$ is thus given by (8) with $\{L_n\}$ and $\{M_n\}$ appeared in (4). Then, the integrability of the AL equation (2) is just equivalent to the vanishing of the discrete curvature of its connection. Therefore, one may also regard the AL equation (2) as an equation with zero (discrete) curvature representation.

When $\{G_n\}$ is a G-valued sequence depending only on the time variable t, we take the following discrete gauge transformation for a given discrete connection $\{A_n\}$:

$$A_n = (L_n, M_n) \to \widetilde{A}_n = (\widetilde{L}_n, \widetilde{M}_n), \tag{10}$$

to get a new $\{\widetilde{A}_n\}$, where $\widetilde{L}_n = G_{n+1}^{-1} L_n G_n$ and $\widetilde{M}_n = G_n^{-1} M_n G_n - G_n^{-1}\dot{G}_n$. It is very important to find the relationship between the two discrete curvatures F_n^A and $F_n^{\widetilde{A}}$ under the action of the discrete gauge transformation (10).

Lemma 1. *Under the action of the discrete gauge transformation (10), we have* $\forall\, n$

$$F_n^{\widetilde{A}} = G_{n+1}^{-1} F_n^A G_n. \tag{11}$$

Proof. In fact, by the definition, we see

$$
\begin{aligned}
F_n^{\widetilde{A}} &= \dot{\widetilde{L}}_n - \widetilde{M}_{n+1}\widetilde{L}_n + \widetilde{L}_n\widetilde{M}_n \\
&= -G_{n+1}^{-1}\dot{G}_{n+1}G_{n+1}^{-1}L_nG_n + G_{n+1}^{-1}\dot{L}_nG_n + G_{n+1}^{-1}L_n\dot{G}_n \\
&\quad - \left(G_{n+1}^{-1}M_{n+1}G_{n+1} - G_{n+1}^{-1}\dot{G}_{n+1}\right)G_{n+1}^{-1}L_nG_n + G_{n+1}^{-1}L_nG_n\left(G_n^{-1}M_nG_n - G_n^{-1}\dot{G}_n\right) \\
&= G_{n+1}^{-1}(\dot{L}_n - M_{n+1}L_n + L_nM_n)G_n = G_{n+1}^{-1}F_n^AG_n.
\end{aligned}
\tag{12}
$$

This completes the proof of the lemma. □

Now we can express the AL-DNLS equation (1) as a discrete equation with prescribed discrete curvature. For equation (1), let

$$
\begin{aligned}
L_n &= \begin{pmatrix} z & \bar{q}_n z^{-1} \\ -q_n z & z^{-1} \end{pmatrix}, \\
M_n &= \mathrm{i}\begin{pmatrix} 1 - z^2 + z - z^{-1} - \bar{q}_n q_{n-1} + \alpha_n & \beta_n - \bar{q}_n + \bar{q}_{n-1}z^{-2} \\ \gamma_n - q_n + q_{n-1}z^2 & -1 + z^{-2} + z - z^{-1} + q_n\bar{q}_{n-1} + \sigma_n \end{pmatrix}
\end{aligned}
\tag{13}
$$

where z is a free parameter, $\alpha_n, \beta_n, \gamma_n$ and σ_n are independent of z with $\gamma_n = \bar{\beta}_n, \sigma_n = -\bar{\alpha}_n$ and they consist of a matrix

$$
Q_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \sigma_n \end{pmatrix}
$$

which satisfies the following successive relation:

$$
\begin{aligned}
&-Q_{n+1}\begin{pmatrix} 1 & \bar{q}_n \\ -q_n & 1 \end{pmatrix} + \begin{pmatrix} 1 & \bar{q}_n \\ -q_n & 1 \end{pmatrix} Q_n \\
&\quad = \begin{pmatrix} 0 & (\mu - 1)|q_n|^2(\bar{q}_{n+1} + \bar{q}_{n-1}) - \gamma|q_n|^2\bar{q}_n \\ (\mu - 1)|q_n|^2(q_{n+1} + q_{n-1}) - \gamma|q_n|^2 q_n & 0 \end{pmatrix}.
\end{aligned}
\tag{14}
$$

A discrete connection $\{A_n\}$ associated with the AL-DNLS equation (1) is now defined by

$$
A_n = (L_n(t, z), M_n(t, z)),
\tag{15}
$$

where $L_n(t, z) = L_n$ and $M_n(t, z) = M_n$ are given by (13). It is a direct computation by using formula (9) that the discrete curvature $\left\{F_n^A\right\}$ of the connection (15) is

$$
F_n^A = \dot{L}_n - M_{n+1}L_n + L_nM_n = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},
\tag{16}
$$

where

$$
\begin{aligned}
a_{11} &= \mathrm{i}z(-\alpha_{n+1} + q_n\beta_{n+1} + \alpha_n) + \mathrm{i}z^{-1}\bar{q}_n\gamma_n \\
a_{12} &= \mathrm{i}z\beta_n + z^{-1}[\dot{\bar{q}}_n + \mathrm{i}(-\bar{q}_n\alpha_{n+1} - \beta_{n+1} + \bar{q}_{n+1} - 2\bar{q}_n + \bar{q}_{n-1} + |q_n|^2(\bar{q}_{n+1} + \bar{q}_{n-1}) + \bar{q}_n\sigma_n)] \\
a_{21} &= z[-\dot{q}_n + \mathrm{i}(-\gamma_{n+1} + q_n\sigma_{n+1} + q_{n+1} - 2q_n + q_{n-1} + |q_n|^2(q_{n+1} + q_{n-1}) - q_n\alpha_n)] + \mathrm{i}z^{-1}\gamma_n \\
a_{22} &= -\mathrm{i}zq_n\beta_n + \mathrm{i}z^{-1}(-\gamma_{n+1}\bar{q}_n - \sigma_{n+1} + \sigma_n).
\end{aligned}
$$

Let

$$
K_n = \mathrm{i}(z - z^{-1})\begin{pmatrix} -\bar{q}_n\gamma_n & \beta_n \\ -\gamma_n & -q_n\beta_n \end{pmatrix}
$$

be a prescribed discrete curvature. Then, it is a straightforward verification, by using (14), that the AL-DNLS equation (1) is exactly equivalent to holding the following prescribed discrete curvature representation:

$$F_n^A = K_n. \tag{17}$$

When $\gamma = 0$ and $\mu = 1$ (i.e. the AL equation (2)), (14) can be solved explicitly by $\alpha_n = \beta_n = \gamma_n = \sigma_n = 0$ and consequently $K_n \equiv 0$. It reduces to zero curvature representation, as mentioned above.

From now on, we restrict ourselves to equation (1) with $\mu > 0$. The discussion of the case $\mu \leqslant 0$ is similar. In this case, by a scaling we may normalize equation (1) into the following form:

$$\mathrm{i}\dot{q}_n + (q_{n+1} + q_{n-1} - 2q_n) + |q_n|^2(q_{n+1} + q_{n-1}) - \hat{\gamma}|q_n|^2 q_n = 0, \tag{18}$$

where $\hat{\gamma} = \gamma/\mu$. We will denote $\hat{\gamma}$ by γ in the following. Assume $\{q_n\}$ is a solution to equation (18), from its prescribed discrete curvature representation (17) (with $\mu = 1$) we see that the discrete connection $\{A_n = (L_n(t,z), M_n(t,z))\}$ defined by (15) with $\mu = 1$ has zero discrete curvature at $z = 1$. This is equivalent to that the following system

$$G_{n+1} = L_n(t,1)G_n, \qquad \dot{G}_n = M_n(t,1)G_n, \quad \forall n \tag{19}$$

is solvable. Let $\{G_n\}$ be a fundamental solution to (19) which is of the form $\begin{pmatrix} g_n & p_n \\ \bar{p}_n & -\bar{g}_n \end{pmatrix}$ and we thus use it to make the following discrete gauge transformation:

$$A_n = (L_n, M_n) \to \widetilde{A}_n = (\widetilde{L}_n, \widetilde{M}_n) := \left(G_{n+1}^{-1} L_n G_n, G_n^{-1} M_n G_n - G_n^{-1}\dot{G}_n\right). \tag{20}$$

Now we deduce explicit expressions for the two components $\widetilde{L}_n, \widetilde{M}_n$ of the discrete connection $\{\widetilde{A}_n\}$ in (20) and calculate the discrete curvature $\left\{F_n^{\widetilde{A}}\right\}$. First,

$$\begin{aligned}
\widetilde{L}_n &= G_{n+1}^{-1} L_n G_n = G_{n+1}^{-1}\begin{pmatrix} z & \bar{q}_n z^{-1} \\ -q_n z & z^{-1} \end{pmatrix} G_n \\
&= G_{n+1}^{-1}\left[\frac{z+z^{-1}}{2}\begin{pmatrix} 1 & \bar{q}_n \\ -q_n & 1 \end{pmatrix} + \frac{z-z^{-1}}{2}\begin{pmatrix} 1 & \bar{q}_n \\ -q_n & 1 \end{pmatrix}\sigma_3\right] G_n \\
&= \frac{z+z^{-1}}{2} I + \frac{z-z^{-1}}{2} G_n^{-1}\sigma_3 G_n \\
&:= \frac{z+z^{-1}}{2} I + \frac{z-z^{-1}}{2} S_n
\end{aligned} \tag{21}$$

where I denotes the 2×2 unit matrix and

$$S_n = G_n^{-1}\sigma_3 G_n. \tag{22}$$

Next,

$$\begin{aligned}
\widetilde{M}_n &= G_n^{-1} M_n G_n - G_n^{-1}\dot{G}_n = G_n^{-1}\left[M_n(t,z) - M_n(t,1)\right] G_n \\
&= \mathrm{i}G_n^{-1}\begin{pmatrix} 1 - z^2 + z - z^{-1} & (z^{-2}-1)\bar{q}_{n-1} \\ (z^2-1)q_{n-1} & -1 + z^{-2} + z - z^{-1} \end{pmatrix} G_n \\
&= \mathrm{i}(z - z^{-1})I + \mathrm{i}\left(\frac{z^2+z^{-2}}{2} - 1\right) G_n^{-1}\begin{pmatrix} -1 & \bar{q}_{n-1} \\ q_{n-1} & 1 \end{pmatrix} G_n \\
&\quad + \mathrm{i}\frac{z^2 - z^{-2}}{2} G_n^{-1}\begin{pmatrix} -1 & -\bar{q}_{n-1} \\ q_{n-1} & -1 \end{pmatrix} G_n \\
&= \mathrm{i}(z - z^{-1})I + \mathrm{i}\left(1 - \frac{z^2+z^{-2}}{2}\right)\frac{S_n + S_{n-1}}{1 + \frac{1}{2}\operatorname{tr}(S_n S_{n-1})} - \mathrm{i}\frac{z^2 - z^{-2}}{2}\frac{I + S_{n-1}S_n}{1 + \frac{1}{2}\operatorname{tr}(S_n S_{n-1})}.
\end{aligned} \tag{23}$$

Here, we have used the identities

$$1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)=\frac{2}{1+|q_n|^2}, \tag{24}$$

$$G_n^{-1}\begin{pmatrix}1 & \bar{q}_{n-1}\\ -q_{n-1} & 1\end{pmatrix}G_n=G_{n-1}^{-1}G_n=\frac{I+S_{n-1}S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n-1}S_n)} \tag{25}$$

and

$$G_n^{-1}\begin{pmatrix}1 & -\bar{q}_{n-1}\\ -q_{n-1} & -1\end{pmatrix}G_n=\frac{S_n+S_{n-1}}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})} \tag{26}$$

which are deduced from (19) and (22) (see the appendix in the end of the paper for the detailed proofs of them). Finally, we come to compute the discrete curvature $\{F_n^{\widetilde{A}}\}$. On the one hand, from the definition we have

$$\begin{aligned}
F_n^{\widetilde{A}} &= \dot{\widetilde{L}}_n-\widetilde{M}_{n+1}\widetilde{L}_n+\widetilde{L}_n\widetilde{M}_n\\
&=\frac{z-z^{-1}}{2}\dot{S}_n-\mathrm{i}\left(1-\frac{z^2+z^{-2}}{2}\right)\frac{z+z^{-1}}{2}\frac{S_{n+1}+S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}\\
&\quad+\mathrm{i}\frac{z^2-z^{-2}}{2}\frac{z+z^{-1}}{2}\frac{I+S_nS_{n+1}}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}-\mathrm{i}\left(1-\frac{z^2+z^{-2}}{2}\right)\frac{z-z^{-1}}{2}\frac{(S_{n+1}+S_n)S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}\\
&\quad+\mathrm{i}\frac{z^2-z^{-2}}{2}\frac{z-z^{-1}}{2}\frac{(I+S_nS_{n+1})S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}+\mathrm{i}\left(1-\frac{z^2+z^{-2}}{2}\right)\frac{z+z^{-1}}{2}\frac{S_n+S_{n-1}}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})}\\
&\quad-\mathrm{i}\frac{z^2-z^{-2}}{2}\frac{z+z^{-1}}{2}\frac{I+S_{n-1}S_n}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})}+\mathrm{i}\left(1-\frac{z^2+z^{-2}}{2}\right)\frac{z-z^{-1}}{2}\frac{S_n(S_n+S_{n-1})}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})}\\
&\quad-\mathrm{i}\frac{z^2-z^{-2}}{2}\frac{z-z^{-1}}{2}\frac{S_n(I+S_{n-1}S_n)}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})}\\
&=\frac{z-z^{-1}}{2}\left(\dot{S}_n-\frac{i[S_{n+1},S_n]}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}+\frac{i[S_n,S_{n-1}]}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})}\right),
\end{aligned} \tag{27}$$

where the proof of the last step of equation (27) is given in the appendix at the end of the paper. On the other hand, by lemma 1, we see that this discrete curvature $\{F_n^{\widetilde{A}}\}$ should also be

$$\begin{aligned}
F_n^{\widetilde{A}} &= G_{n+1}^{-1}F_n^AG_n=G_{n+1}^{-1}K_nG_n=\mathrm{i}(z-z^{-1})G_{n+1}^{-1}\begin{pmatrix}-\bar{q}_n\gamma_n & \beta_n\\ -\gamma_n & -q_n\beta_n\end{pmatrix}G_n\\
&=\mathrm{i}(z-z^{-1})G_n^{-1}\begin{pmatrix}0 & \beta_n\\ -\gamma_n & 0\end{pmatrix}G_n=\mathrm{i}\frac{z-z^{-1}}{2}G_n^{-1}(\sigma_3Q_n-Q_n\sigma_3)G_n\\
&=\mathrm{i}\frac{z-z^{-1}}{2}\left(G_n^{-1}\sigma_3G_nG_n^{-1}Q_nG_n-G_n^{-1}Q_nG_nG_n^{-1}\sigma_3G_n\right)\\
&=\mathrm{i}\frac{z-z^{-1}}{2}\left[S_n,G_n^{-1}Q_nG_n\right].
\end{aligned} \tag{28}$$

We come to express the right-hand side of (28) in terms of $\{S_n\}$. From (14) with $\mu=1$, we see

$$-G_{n+1}^{-1}Q_{n+1}G_{n+1}+G_n^{-1}Q_nG_n=G_{n+1}^{-1}\begin{pmatrix}0 & -\gamma|q_n|^2\bar{q}_n\\ -\gamma|q_n|^2q_n & 0\end{pmatrix}G_n$$

$$
\begin{aligned}
&= -\gamma G_n^{-1}\frac{|q_n|^2}{1+|q_n|^2}\begin{pmatrix}1 & -\bar{q}_n\\ q_n & 1\end{pmatrix}\begin{pmatrix}0 & \bar{q}_n\\ q_n & 0\end{pmatrix}G_n\\
&= -\gamma\frac{|q_n|^2}{1+|q_n|^2}\left[-|q_n|^2G_n^{-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}G_n + G_n^{-1}\begin{pmatrix}0 & \bar{q}_n\\ q_n & 0\end{pmatrix}G_n\right]\\
&= -\gamma\frac{|q_n|^2}{1+|q_n|^2}\left[-|q_n|^2S_n + G_n^{-1}\begin{pmatrix}1 & \bar{q}_n\\ q_n & -1\end{pmatrix}G_n - S_n\right]\\
&= \gamma\frac{1-\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}S_n - \frac{\gamma}{2}\frac{1-\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}(S_{n+1}+S_n)\\
&= \gamma\left(\frac{1}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)} - \frac{1}{2}\right)(S_n - S_{n+1}).
\end{aligned}
\tag{29}
$$

Here in the last third equality we have used (22) and in the last second equality we have used the identities

$$
1-\frac{1}{2}\operatorname{tr}(S_{n+1}S_n) = 2|q_n|^2/(1+|q_n|^2), \qquad 1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n) = \frac{2}{1+|q_n|^2} \tag{30}
$$

and

$$
G_n^{-1}\begin{pmatrix}1 & \bar{q}_n\\ q_n & -1\end{pmatrix}G_n = \frac{S_{n+1}+S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}. \tag{31}
$$

These identities are also proved explicitly in the appendix. From (29), we obtain

$$
G_n^{-1}Q_nG_n - G_{n-1}^{-1}Q_{n-1}G_{n-1} = \gamma\left(\frac{1}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})} - \frac{1}{2}\right)(S_n - S_{n-1}). \tag{32}
$$

From now on, we fix $Q_0 = 0$ in solving the successive equation (14) with $\mu = 1$. Thus, the right-hand side of (28) is $\mathrm{i}\frac{z-z^{-1}}{2}[S_n, P_n]$ with P_n being given by

$$
P_n = \begin{cases}
\sum_{k=1}^{n-1}\left(-\frac{\gamma}{1+\frac{1}{2}\operatorname{tr}(S_{k+1}S_k)} + \frac{\gamma}{1+\frac{1}{2}\operatorname{tr}(S_kS_{k-1})}\right)S_k + \left(\frac{\gamma}{1+\frac{1}{2}\operatorname{tr}(S_1S_0)} - \frac{\gamma}{2}\right)S_0, & n>0,\\
0, & n=0,\\
\sum_{k=-1}^{n+1}\left(-\frac{\gamma}{1+\frac{1}{2}\operatorname{tr}(S_kS_{k-1})} + \frac{\gamma}{1+\frac{1}{2}\operatorname{tr}(S_{k+1}S_k)}\right)S_k + \left(\frac{\gamma}{1+\frac{1}{2}\operatorname{tr}(S_0S_{-1})} - \frac{\gamma}{2}\right)S_0, & n<0.
\end{cases} \tag{33}
$$

Equating (27) and (28) (since lemma 1), we see that $\{S_n\}$ satisfies the following integrable–nonintegrable discrete Heisenberg model:

$$
\frac{\mathrm{d}S_n}{\mathrm{d}t} = \frac{\mathrm{i}[S_{n+1}, S_n]}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)} - \frac{\mathrm{i}[S_n, S_{n-1}]}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})} + \mathrm{i}[S_n, P_n]. \tag{34}
$$

Remark 1. We would like to point out that, from equation (19), the gauge function G_n is in general of the form $\left(\begin{smallmatrix}g_n & p_n\\ \bar{p}_n & -\bar{g}_n\end{smallmatrix}\right)$. Thus, the matrix S_n given by $G_n^{-1}\sigma_3G_n$ is of the form $\left(\begin{smallmatrix}s_n^1 & s_n^2-\mathrm{i}s_n^3\\ s_n^2+\mathrm{i}s_n^3 & -s_n^1\end{smallmatrix}\right)$ for some s_n^1, s_n^2, s_n^3 with $\mathbf{S}_n = (s_n^1, s_n^2, s_n^3) \in \mathbf{S}^2 \hookrightarrow \mathbf{R}^3$ (since $S_n^2 = G_n^{-1}\sigma_3G_nG_n^{-1}\sigma_3G_n = I$). Furthermore, equation (34) is equivalent to

$$
\frac{\mathrm{d}\mathbf{S}_n}{\mathrm{d}t} = -2\frac{\mathbf{S}_n\times\mathbf{S}_{n+1}}{1+(\mathbf{S}_n\cdot\mathbf{S}_{n+1})} - 2\frac{\mathbf{S}_n\times\mathbf{S}_{n-1}}{1+(\mathbf{S}_n\cdot\mathbf{S}_{n-1})} + 2\mathbf{S}_n\times\mathbf{P}_n, \tag{35}
$$

in which $\mathbf{P}_n$ is given by (33) only with matrices S_k being replaced by vectors $\mathbf{S}_k$ and $\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)$ being replaced by $\mathbf{S}_n\cdot\mathbf{S}_{n+1}$.

It is obvious that (34) is an equation with prescribed discrete curvature representation:

$$F_n^{\widetilde{A}} = \widetilde{K}_n, \tag{36}$$

where $\{\widetilde{A}_n = (\widetilde{L}_n, \widetilde{M}_n)\}$ with $\widetilde{L}_n$ being given by (21) and $\widetilde{M}_n$ by (23), and $\widetilde{K}_n = \mathrm{i}\frac{z-z^{-1}}{2}[S_n, P_n]$. In fact, since we have fixed $Q_0 = 0$, obviously $G_0^{-1}Q_0G_0 = 0$. For $n > 0$, replacing the subscript n with k in (32) and summing equation (32) with respect to k from 1 to n, we obtain $G_n^{-1}Q_nG_n = P_n$. Similarly, we also obtain $G_n^{-1}Q_nG_n = P_n$ in the case of $n < 0$. This establishes (36) by applying (28).

To sum up, we have

Theorem 1. *The AL-DNLS equation (18) is discrete gauge equivalent to the integrable–nonintegrable discrete Heisenberg model (34) by the gauge matric sequence $\{G_n\}$ satisfying (19). Especially, when $\gamma = 0$, the result is reduced to the known fact that the AL equation of focusing type is gauge equivalent to the DHM.*

3. Applications

It is well known that, in integrable case, N-soliton solutions to the AL equation (2) of focusing (resp. de-focusing) type correspond to N-soliton solutions to the integrable discrete Heisenberg spin model (6) (resp. (7)) under the action of discrete gauge transformations [11–13, 16]. Furthermore, it is also proved that these discrete gauge equivalences also fulfil the corresponding principle in quantum theory [13]. This shows that solitonic properties are preserved under the action of discrete gauge transformations. As we have established the discrete gauge equivalence between the AL-DNLS equation (18) and the integrable–nonintegrable discrete Heisenberg model (34), it is very interesting and important to see whether chaotic dynamical properties of the AL-DNLS equation (18) are transformed to those of the integrable–nonintegrable discrete Heisenberg model (34) by discrete gauge transformations or not. We only check this here for the transmission and bifurcation-creating properties of the AL-DNLS equation (18). One knows that these properties are related to the discrete KAM theory and quantum chaotic properties of the AL-DNLS equation (18) (see, for example, [8, 6]). Since we do not deal with the integrable case (i.e. $\gamma = 0$) in this section, for the sake of simplicity, (34) is called the nonintegrable discrete Heisenberg model and denoted by the N-DHM from now on.

Before doing this, let us give a general description of constructing solutions to the N-DHM (34) from those to the AL-DNLS equation (18). For a solution $\{q_n\}$ to equation (18), as we have chosen $Q_0 = 0$ in getting (34), let G_0 be a fundamental solution to

$$\frac{\mathrm{d}G_0}{\mathrm{d}t} = M_0(t,1)G_0 = \mathrm{i}\begin{pmatrix} -\bar{q}_0q_{-1} & -\bar{q}_0 + \bar{q}_{-1} \\ -q_0 + q_{-1} & q_0\bar{q}_{-1} \end{pmatrix} G_0. \tag{37}$$

Using (14) with $\mu = 1$, it is a direct verification that, by successive iteration,

$$G_n = \begin{cases} L_{n-1}(t,1)G_{n-1}, & n > 0 \\ L_{n+1}(t,1))^{-1}G_{n+1}, & n < 0 \end{cases} = \begin{cases} L_{n-1}(t,1)\cdots L_0(t,1)G_0, & n > 0 \\ (L_{n+1}(t,1)\cdots L_0(t,1))^{-1}G_0, & n < 0 \end{cases}$$

$$= \begin{cases} \begin{pmatrix} A_n & B_n \\ -\bar{B}_n & \bar{A}_n \end{pmatrix} G_0, & n > 0 \\ \begin{pmatrix} A_n & B_n \\ -\bar{B}_n & \bar{A}_n \end{pmatrix}^{-1} G_0, & n < 0 \end{cases} \tag{38}$$

solves equation (19), where for $n > 0$

$$A_{\pm n} = 1 + \sum_{k=1}^{n-1} (-1)^k \sum_{0 \leqslant j_1 < \cdots < j_{2k} \leqslant n-1} \bar{q}_{\pm j_{2k}} q_{\pm j_{2k-1}} \cdots \bar{q}_{\pm j_2} q_{\pm j_1}, \tag{39}$$

$$B_{\pm n} = \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{0 \leqslant j_1 < \cdots < j_{2k-1} \leqslant n-1} \bar{q}_{\pm j_{2k-1}} q_{\pm j_{2k-2}} \cdots \bar{q}_{\pm j_1}. \tag{40}$$

Therefore,

$$S_n = G_n^{-1} \sigma_3 G_n = \begin{cases} \dfrac{1}{|A_n|^2 + |B_n|^2} G_0^{-1} \begin{pmatrix} |A_n|^2 - |B_n|^2 & 2\bar{A}_n B_n \\ 2A_n \bar{B}_n & |B_n|^2 - |A_n|^2 \end{pmatrix} G_0, & n > 0 \\ G_0^{-1} \sigma_3 G_0, & n = 0 \\ \dfrac{1}{|A_n|^2 + |B_n|^2} G_0^{-1} \begin{pmatrix} |A_n|^2 - |B_n|^2 & -2A_n B_n \\ -2\bar{A}_n \bar{B}_n & |B_n|^2 - |A_n|^2 \end{pmatrix} G_0, & n < 0 \end{cases} \tag{41}$$

is a solution to the N-DHM (34) which corresponds to $\{q_n\}$ under the discrete gauge transformation.

3.1. Transmission properties

In this subsection, we study as a physical application whether the wave transmission properties of the nonlinear lattice chain of AL-DNLS equation are preserved under the action of discrete gauge transformations to that of the N-DHM (34). In order to do this, we need to clear what is the meaning of the transmission properties of a nonlinear matrix chain embedded in the nonlinear N-DHM (34) and so do some preliminaries first.

By setting $q_n(t) = \varphi_n \exp[\mathrm{i}(E-2)t]$, where φ_n is a complex amplitude which is independent of the time variable t and E is a real parameter, we get the following recurrence equation originated from the stationary discrete nonlinear Schrödinger equation of the AL-DNLS equation (18):

$$E\varphi_n - (1 + |\varphi_n|^2)(\varphi_{n+1} + \varphi_{n-1}) + \gamma |\varphi_n|^2 \varphi_n = 0,$$

which can be rewritten as

$$\varphi_{n+1} + \varphi_{n-1} = \frac{E + \gamma |\varphi_n|^2}{1 + |\varphi_n|^2} \varphi_n. \tag{42}$$

It is obvious that (42) reduces to a degenerate linear map if $\gamma = E$. Although one may choose E to be $E = \gamma$, we are interested in the nondegenerate case $E \neq \gamma$ in the following. Equation (42) gives a recurrence relation $\varphi_{n+1} = \varphi_{n+1}(\varphi_n, \varphi_{n-1})$ acting as a four-dimensional mapping $\mathbf{C}^2 \to \mathbf{C}^2$. However, this equation can be reduced to a two-dimensional mapping on the plane $\mathbf{R}^2 \to \mathbf{R}^2$. Following Wan and Soukoulis [9], we use polar coordinates for φ_n, that is $\varphi_n = r_n \exp(\mathrm{i}\theta_n)$, and rewrite equation (42) equivalently as

$$r_{n+1} \cos(\Delta\theta_{n+1}) + r_{n-1} \cos(\Delta\theta_n) = \frac{E + \gamma r_n^2}{1 + r_n^2} r_n, \tag{43}$$

$$r_{n+1} \sin(\Delta\theta_{n+1}) - r_{n-1} \sin(\Delta\theta_n) = 0, \tag{44}$$

where $\Delta\theta_n = \theta_n - \theta_{n-1}$. Equation (44) is equivalent to conservation of the probability current

$$J = r_n r_{n-1} \sin(\Delta\theta_n). \tag{45}$$

We further introduce real-valued variables defined by

$$\begin{cases} x_n := \bar{\varphi}_n\varphi_{n-1} + \varphi_n\bar{\varphi}_{n-1} = 2r_n r_{n-1}\cos(\Delta\theta_n), \\ y_n := \mathrm{i}[\bar{\varphi}_n\varphi_{n-1} - \varphi_n\bar{\varphi}_{n-1}] = 2J, \\ z_n := |\varphi_n|^2 - |\varphi_{n-1}|^2 = r_n^2 - r_{n-1}^2. \end{cases} \tag{46}$$

Then, the system of equations (43), (44) can be rewritten as a two-dimensional real map $M_{\gamma,E}$ as follows:

$$M_{\gamma,E} : \begin{cases} x_{n+1} = \dfrac{E + \frac{1}{2}\gamma(w_n + z_n)}{1 + \frac{1}{2}(w_n + z_n)}(w_n + z_n) - x_n \\ z_{n+1} = \dfrac{1}{2}\dfrac{x_{n+1}^2 - x_n^2}{w_n + z_n} - z_n, \end{cases} \tag{47}$$

where $w_n = \sqrt{x_n^2 + z_n^2 + 4J^2}$.

Let us review the transmission properties of the AL-DNLS lattice chain (see, for example, [6, 8] for details). We consider a piece of nonlinear medium material $0 \leqslant x \leqslant N$ of length N in a stationary regime. An incident plane waves $R_0\,\mathrm{e}^{\mathrm{i}kx}$ on the left ($x \leqslant 0$) induces a reflected plane wave $R\,\mathrm{e}^{-\mathrm{i}kx}$ on the left and a transmitted plane wave $T\,\mathrm{e}^{\mathrm{i}kx}$ on the right ($x \geqslant N$). The transmission problem associated with (42) reads

$$\varphi_n = \begin{cases} R_0\,\mathrm{e}^{\mathrm{i}kn} + R\,\mathrm{e}^{-\mathrm{i}kn}, & -1 \leqslant n \leqslant N \\ T\,\mathrm{e}^{\mathrm{i}kn}, & n \geqslant N. \end{cases} \tag{48}$$

We denote by R_0, R the amplitudes of the incoming and reflected waves and by T the transmitted amplitude at the right end of the nonlinear chain. $|R_0|^2$ and $|T|^2$ are called the incoming wave intensity and the transmitted intensity, respectively. R and T depend on the wave number k and on N. The medium is nonlinear, thus the transmission coefficient as a function of the incoming intensity $|R_0|^2$ is not a constant. Then, for a given value of the incoming intensity, there may be several value of $|R|^2$ and $|T|^2$, this is called bistability [6]. However, for a given k, we can solve (42) step by step from $n = N$ to $n = 0$ for an output $T\,\mathrm{e}^{\mathrm{i}kn}$ and then find R_0 and R. Thus, the pair $(k, |T|)$ initializes the incoming amplitude R_0 completely (see, for example, [8]). In fact, we have $(\varphi_{N+1}, \varphi_N) = (T\,\mathrm{e}^{[\mathrm{i}k(N+1)]}, T\,\mathrm{e}^{[\mathrm{i}kN]})$. From the pair $(\varphi_{N+1}, \varphi_N)$ we get (r_{N+1}, r_N) and (θ_{N+1}, θ_N) as well as $x_{N+1} = 2|T|\cos k$, $z_{N+1} = 0$. Using equations (43) and (44) and iterating from $n = N$ to 0 successively determines the amplitudes $(r_{N-1}, \ldots, r_0)$ and phases $(\theta_{N-1}, \ldots, \theta_0)$. Thus, we eventually obtain the value of φ_0 and hence R_0 and R on the left end of the nonlinear chain.

If the resulting incoming wave intensity $|R_0|^2$ is of the same order of the transmitted intensity $|T|^2$, independent of N, we say that the nonlinear chain with wave number k and outgoing intensity $|T|^2$ is to be transmitting. If R_0 appears to be a rapidly increasing function of N, we say that this nonlinear chain is to be non-transmitting. Figures 1–3 display the transmission behaviours in the (k, T) parameter plane, representing region of transmitting (white) and non-transmitting (hatched) behaviours. The transmission behaviour of the general AL-DNLS (1) has been displayed in [6, 8], here we only present the case with $\mu = 1$ and $\gamma = 4$ in figure 1, and figures 2 and 3 are enlargements of figure 1 in the local area of $0 \leqslant k \leqslant 0.6, 0.4 \leqslant T \leqslant 2$ and $1.6 \leqslant k \leqslant 1.8, 0 \leqslant T \leqslant 0.3$, respectively. In figure 1, the white regions correspond to values of (k, T) for which the nonlinear chain is to be transmitting. In such a situation, the intensity R_0 remains of the same order as T when N increases. On the other hand, in the hatched regions, the amplitude R_0, as a function of N, is rapidly increasing and thus the nonlinear chain falls in a chaotic region, as also pointed out in [6] in the case of DNLS equation. For wave numbers $k \in \left[0, \frac{\pi}{2}\right]$, the regions of bounded (i.e., transmitting) and

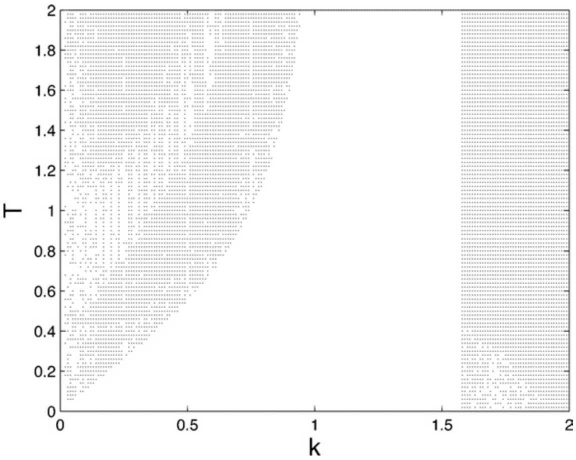

Figure 1. Wave transmission properties in the (k, T) plane. Hatched regions correspond the non-transmitting regime whereas the white regions show transmission. The chain length is $N = 500$ in the case of $\mu = 1$ and $\gamma = 4$.

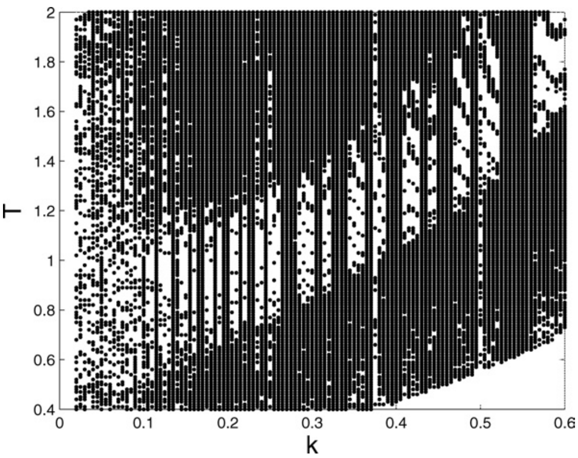

Figure 2. The enlargement of figure 1 in the area $0 \leqslant k \leqslant 0.6$, $0.4 \leqslant T \leqslant 2$.

unbounded (i.e., non-transmitting) solutions are separated by a sharp smooth curve as shown in figure 1. Interesting feature appears in figure 3, which shows that the transmitting regions consist a sequence of fractal crescent shapes in the area of $1.6 \leqslant k \leqslant 1.8$, $0 \leqslant T \leqslant 0.3$.

The transmission problem associated with the nonintegrable discrete Heisenberg model (34) is now proposed as follows. There is a finite nonlinear matrix wave chain $\{S_n = G_n^{-1}\sigma_3 G_n\}$ $(1 \leqslant n \leqslant N)$ embedded in a nonlinear chain of the N-DHM (34) from the left towards the right, where they are scattered into reflected and transmitted parts, which look like

$$G_n = \begin{pmatrix} 1 & \bar{q}_{n-1} \\ -q_{n-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \bar{q}_0 \\ -q_0 & 1 \end{pmatrix} G_0, \qquad n \geqslant 1, \tag{49}$$

with G_0 satisfying (37) and q_n being given by

$$q_n = \begin{cases} e^{i(E-2)t}(R_0\, e^{-ikn} + R\, e^{ikn}), & -1 \leqslant n \leqslant N \\ e^{i(E-2)t} T\, e^{-ikn}, & n \geqslant N. \end{cases} \tag{50}$$

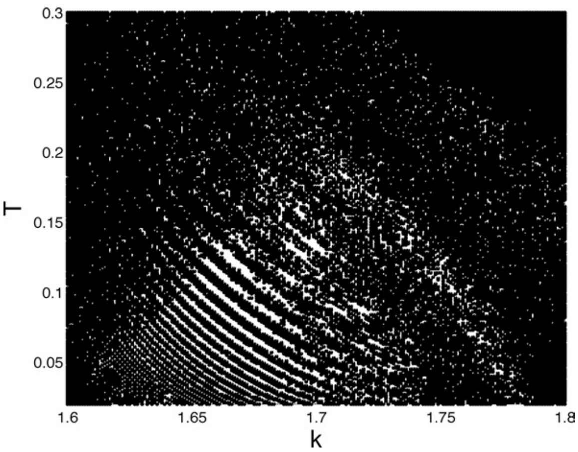

Figure 3. The enlargement of figure 1 in the area $1.6 \leqslant k \leqslant 1.8, 0 \leqslant T \leqslant 0.3$. The white layer regions of crescent shape indicate the fractal phenomenon in this area.

Similarly, R_0, R present the amplitudes of the incoming and reflected waves and T the transmitted amplitude at the right end of the nonlinear chain under consideration. $|R_0|^2$ and $|T|^2$ are also called the incoming wave intensity and the transmitted intensity of the nonlinear matrix chain, respectively.

In order to display the transmission behaviour of the nonlinear matrix chain $\left\{S_n = G_n^{-1}\sigma_3 G_n\right\}$ with G_n given by (49), we must first show that the pair (k, T) initializes the incoming amplitude R_0 completely. This is done by the following lemma.

Lemma 2. *Assume that the matrix nonlinear chain* $\left\{S_n = G_n^{-1}\sigma_3 G_n\right\}$ *with* G_n *given by (49) satisfies the N-DHM (34) for* $n \geqslant 1$*. Then, the plane waves* $\{q_n\}$ *given by (50) for* $n \geqslant -1$ *satisfies the AL-DNLS equation (18) up to a factor of the form* $\mathrm{e}^{2\mathrm{i}\alpha_0}$*, where* α_0 *is a real number.*

Proof. For the matrix sequence $\{S_n\}$ given in the lemma, we shall find a sequence $\{\widetilde{G}_n\}$ such that $\sigma_3 = \widetilde{G}_n S_n \widetilde{G}_n^{-1}$ and

$$\widetilde{G}_{n+1} = \begin{pmatrix} 1 & \bar{\widetilde{q}}_n \\ -\widetilde{q}_n & 1 \end{pmatrix} \widetilde{G}_n \tag{51}$$

for some complex $\widetilde{q}_n$. In fact, the general solution to $\sigma_3 = \widetilde{G}_n S_n \widetilde{G}_n^{-1}$ is (see, for example, [16])

$$\widetilde{G}_n = \mathrm{diag}(F_n, \bar{F}_n) G_n, \tag{52}$$

where F_n is a complex number. Because of the requirement of (51), we see that $F_{n+1} = F_n = \cdots = F_0$ by substituting (52) into (51) and hence $\widetilde{q}_n = F_{n+1} q_n \bar{F}_n^{-1} = \mathrm{e}^{\mathrm{i}2\alpha_0} q_n$ for some real α_0 (when we write $F_0 = r\,\mathrm{e}^{\mathrm{i}\alpha_0}$ for some $r > 0$ and α_0). Put

$$L_n^G = \widetilde{G}_{n+1} \widetilde{L}_n(z) \widetilde{G}_n^{-1} = \begin{pmatrix} z & \bar{\widetilde{q}}_n z^{-1} \\ -\widetilde{q}_n z & z^{-1} \end{pmatrix},$$

$$M_n^G = \dot{\widetilde{G}}_n \widetilde{G}_n^{-1} + \widetilde{G}_n \widetilde{M}_n(z) \widetilde{G}_n^{-1} = \dot{\widetilde{G}}_n \widetilde{G}_n^{-1} + \mathrm{i} \begin{pmatrix} 1 - z^2 + z - z^{-1} & \bar{\widetilde{q}}_{n-1}(z^{-2} - 1) \\ \widetilde{q}_{n-1}(z^2 - 1) & -1 + z^{-2} + z - z^{-1} \end{pmatrix},$$

where $\widetilde{L}_n$ and $\widetilde{M}_n$ are given by (21) and (23), respectively, where $\{S_n\}$ are given as above. Since $\{S_n\}$ satisfies the N-DHM (34) or, equivalently, $\widetilde{L}_n$ and $\widetilde{M}_n$ satisfy the prescribed discrete curvature condition (36), by lemma 1 we have

$$\dot{L}_n^G + L_n^G M_n^G - M_{n+1}^G L_n^G = \widetilde{G}_{n+1}\widetilde{K}_n\widetilde{G}_n^{-1}, \tag{53}$$

where $\widetilde{K}_n = \mathrm{i}\frac{z-z^{-1}}{2}[S_n, P_n]$ with P_n given by (33). It is a direct computation that

$$\text{lhs of (53)} = \begin{pmatrix} 1 & \tilde{\tilde{q}}_n \\ -\tilde{q}_n & 1 \end{pmatrix}\left[\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\dot{\widetilde{G}}_n\widetilde{G}_n^{-1} - \dot{\widetilde{G}}_n\widetilde{G}_n^{-1}\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\right]$$
$$+\mathrm{i}(z-z^{-1})\begin{pmatrix} \tilde{\tilde{q}}_n\tilde{q}_{n-1} - |q_n|^2 & \tilde{\tilde{q}}_n - \tilde{\tilde{q}}_{n-1} \\ -\tilde{q}_n + \tilde{q}_{n-1} & \tilde{q}_n\tilde{\tilde{q}}_{n-1} - |\tilde{q}_n|^2 \end{pmatrix}$$

$$\text{rhs of (53)} = \mathrm{i}(z-z^{-1})\begin{pmatrix} -\tilde{\tilde{q}}_n\gamma_n & \beta_n \\ -\gamma_n & -\tilde{q}_n\beta_n \end{pmatrix},$$

where β_n and γ_n are off-diagonal entries of the matrix Q_n satisfying (14) (in which $\{q_n\}$ is just replaced by $\{\tilde{q}_n\}$) with $\mu = 1$. Thus, by substituting the above expressions into (53), the equation of the off-diagonal part of (53) leads to

$$\text{off-diagonal of } \dot{\widetilde{G}}_n\widetilde{G}_n^{-1} = \mathrm{i}\begin{pmatrix} * & -\tilde{\tilde{q}}_n + \tilde{\tilde{q}}_{n-1} + \beta_n \\ -\tilde{q}_n + \tilde{q}_{n-1} + \gamma_n & * \end{pmatrix}, \tag{54}$$

where $*$ are some expressions which cannot be carried out at this moment. On the other hand, at the same time we also have

$$\text{lhs of (53)} = \begin{pmatrix} 0 & \dot{\tilde{\tilde{q}}}_n z^{-1} \\ -\dot{\tilde{q}}_n z & 0 \end{pmatrix} + \begin{pmatrix} z & \tilde{\tilde{q}}_n z^{-1} \\ -\tilde{q}_n z & z^{-1} \end{pmatrix}\dot{\widetilde{G}}_n\widetilde{G}_n^{-1} - \dot{\widetilde{G}}_{n+1}\widetilde{G}_{n+1}^{-1}\begin{pmatrix} z & \tilde{\tilde{q}}_n z^{-1} \\ -\tilde{q}_n z & z^{-1} \end{pmatrix}$$
$$+\mathrm{i}(z-z^{-1})\begin{pmatrix} \tilde{\tilde{q}}_n\tilde{q}_{n-1} - |\tilde{q}_n|^2 & \tilde{\tilde{q}}_n - \tilde{\tilde{q}}_{n-1} \\ -\tilde{q}_n + \tilde{q}_{n-1} & \tilde{q}_n\tilde{\tilde{q}}_{n-1} - |\tilde{q}_n|^2 \end{pmatrix}.$$

The equation of the diagonal part of (53) in this time implies

$$\text{diagonal of } \dot{\widetilde{G}}_n\widetilde{G}_n^{-1} = \mathrm{i}\begin{pmatrix} -\tilde{\tilde{q}}_n\tilde{q}_{n-1} + \alpha_n & * \\ * & \tilde{q}_n\tilde{\tilde{q}}_{n-1} + \sigma_n \end{pmatrix} + \mathrm{i}\tau(t)\sigma_3, \tag{55}$$

for some real-valued function $\tau(t)$ depends only on t. Combining (54) with (55) we finally obtain

$$\dot{\widetilde{G}}_n\widetilde{G}_n^{-1} = \mathrm{i}\begin{pmatrix} -\tilde{\tilde{q}}_n\tilde{q}_{n-1} + \alpha_n & -\tilde{\tilde{q}}_n + \tilde{\tilde{q}}_{n-1} + \beta_n \\ -\tilde{q}_n + \tilde{q}_{n-1} + \gamma_n & \tilde{q}_n\tilde{\tilde{q}}_{n-1} + \sigma_n \end{pmatrix} + \mathrm{i}\tau(t)\sigma_3. \tag{56}$$

Note that the above restriction on $\widetilde{G}_n$ allows an arbitrariness in $\widetilde{G}_n$ of the form $\widetilde{G}_n \to \mathrm{e}^{\mathrm{i}\kappa(t)\sigma_3}\widetilde{G}_n$ for a real-valued function $\kappa(t)$. If we require $\kappa(t)$ to fulfil $\dot{\kappa}(t) = -\tau(t)$, then $\widetilde{G}_n$ can be modified so that for the new $\widetilde{G}_n$ the second term on the right-hand side of (56) is zero. This indicates that $L_n^G(z)$ and $M_n^G(z)$ are exactly the two components of the discrete connection of the AL-DNLS equation (18) and hence $\hat{q}_n = \mathrm{e}^{2\mathrm{i}(\alpha_0+k(t))}q_n$ satisfies the AL-DNLS equation (18). However, by substituting $\hat{q}_n = \mathrm{e}^{2\mathrm{i}(\alpha_0+k(t))}q_n$ into (18), we see $\dot{\kappa}(t) = 0$ (since q_n is nontrivial), i.e., $\kappa(t) =$ real const. This proves lemma 2. □

We remark that lemma 2 actually shows that the inverse of theorem 1 is still true. Lemma 2 tells us that $\hat{q}_n = \mathrm{e}^{2\mathrm{i}\alpha}q_n$ satisfies the AL-DNLS equation (18) for some real number α and so does q_n. This implies that

$$\varphi_n = \begin{cases} (R_0\,\mathrm{e}^{\mathrm{i}kn} + R\,\mathrm{e}^{-\mathrm{i}kn}), & -1 \leqslant n \leqslant N \\ T\,\mathrm{e}^{\mathrm{i}kn}, & n \geqslant N \end{cases} \tag{57}$$

solves equation (42). From the known transmission properties of equation (42) indicated above, we see that the pair (k, T) of the momentum and transmitted amplitude of the nonlinear matrix chain $\left\{S_n = G_n^{-1}\sigma_3 G_n\right\}$ given by (49) initializes the incoming amplitude R_0 completely too. Based on this fact we shall say that the nonlinear matrix chain is said to be transmitting if the incoming wave intensity $|R_0|^2$ is of the same order of the transmitted intensity $|T|^2$, otherwise it is said to be non-transmitting. Thus, the nonlinear matrix chain has the similar transmission behaviours as those of the AL-DNLS equation. Figures 1–3 are completely suitable in displaying the transmission behaviours of the finite nonlinear matrix chain of the N-DHM (34) in the parameters (k, T) plane with chain length $N = 500$ in the case of $\mu = 1$ and $\gamma = 4$. This illustrates not only that the N-DHM (34) has transmission properties, but also that the transmission properties are preserved and delivered under the action of discrete gauge transformations.

Theorem 2. *The wave transmission properties of the nonlinear lattice chain of the AL-DNLS equation (18) are transformed to those of the nonlinear matrix chain of the N-DHM (34) by discrete gauge transformations.*

We would like to point out that theorem 2 reveals an interesting fact that some quantum chaotic properties of different nonlinear discrete equations may relate and interact to each other.

3.2. Period-doubling bifurcation sequence

In this subsection, we first briefly review constructions of period-1 and period-2 solutions to (42) or equivalently the map (47), which indicates the bifurcation-creating property of the AL-DNLS equation (18). Then, we display whether these period solutions are preserved under the action of discrete gauge transformations.

We need to divide our discussion into two different situations. One is the case that the probability current $J \neq 0$. In this case, it is convenient to introduce the scaling transformation: $2Jx_n \to \widetilde{x}_n$, $2Jz_n \to \widetilde{z}_n$, $2Jw_n \to \widetilde{w}_n$, $J\gamma \to \widetilde{\gamma}$ and $J \to \widetilde{\mu}$. Then, we obtain the scaled map

$$M_{\widetilde{\mu},\widetilde{\gamma},E} : \begin{cases} \widetilde{x}_{n+1} = \dfrac{E + \widetilde{\gamma}(\widetilde{w}_n + \widetilde{z}_n)}{1 + \widetilde{\mu}(\widetilde{w}_n + \widetilde{z}_n)}(\widetilde{w}_n + \widetilde{z}_n) - \widetilde{x}_n, \\ \widetilde{z}_{n+1} = \dfrac{1}{2}\dfrac{\widetilde{x}_{n+1}^2 - \widetilde{x}_n^2}{\widetilde{w}_n + \widetilde{z}_n} - \widetilde{z}_n, \end{cases} \tag{58}$$

with $\widetilde{w}_n = \sqrt{\widetilde{x}_n^2 + \widetilde{z}_n^2 + 1}$. As pointed out in [8], the period-1 orbit (solution) is determined by

$$x = \frac{1}{2}\frac{E + \widetilde{\gamma}w}{1 + \widetilde{\mu}w}w, \qquad z = 0, \tag{59}$$

where $w = \sqrt{x^2 + 1}$. The first equation of (59) possesses one real root for $\widetilde{\gamma} = 0$, resulting in a stable fixed point and has either no root or two real roots for $\widetilde{\gamma} > 0$. The two real roots correspond to one hyperbolic and one elliptic fixed point, respectively. Only for $E < 0$, as a map of the parameters $\widetilde{\mu}, \widetilde{\gamma}, E$, a period-doubling bifurcation is created for the map $M_{\widetilde{\mu},\widetilde{\gamma},E}$, where the stable fixed point is converted into an unstable hyperbolic point with reflection accompanied by creation of two additional elliptic points. The period-2 bifurcation for period-1 orbit sets in when $|E|/\widetilde{\gamma} > 1$ $(E < 0)$ and the newborn period-2 orbits (solutions) are

$$x = \pm\left[\left(\frac{E}{\widetilde{\gamma}}\right)^2 - 1\right]^{1/2}, \qquad z = 0. \tag{60}$$

Note that the location of the period-2 orbits for the map $M_{\widetilde{\mu},\widetilde{\gamma},E}$ depends only on the $(\widetilde{\gamma}, E)$. We would like to point out that, at a sufficiently high $|E|$ value, the period-2 orbit also loses stability caused by a next period-doubling bifurcation, which in turn gives rise to the birth of the corresponding period-4 orbit (see [8]).

Another case is $J = 0$. Similarly, for the two-dimensional real map (47) $M_{\gamma,E}$ with $J = 0$, the period-1 solution is the fixed point of the map (47) with $J = 0$, which is

$$x = x_0 = 2\frac{E-2}{2-\gamma} > 0, \qquad z = 0 \tag{61}$$

when the parameters γ and E satisfy $\frac{E-2}{2-\gamma} > 0$ or

$$x = x_0 = -2\frac{E+2}{2+\gamma} < 0, \qquad z = 0 \tag{62}$$

when the parameters satisfy $\frac{E+2}{2+\gamma} > 0$. The period-2 solution is

$$x = x_0 = \pm 2\left|\frac{E}{\gamma}\right|, \qquad z = 0 \tag{63}$$

when the parameters γ and E satisfy $E\gamma < 0$, which creates period-doubling bifurcation for the map $M_{\gamma,E}$ (47) with $J = 0$. Again the location of the period-2 orbits for the map $M_{\gamma,E}$ (47) with $J = 0$ also depends only on (γ, E).

For the above period-1, period-2 orbit solutions (61)–(63) of the map (47) with $J = 0$ and period-1, period-2 orbit solutions (59), (60) of the map (58) with $J \neq 0$, we come to transfer them into solutions to the AL-DNLS equation (18) and to see if their gauge corresponding solutions to the N-DHM (34) have the same period, respectively.

Note that there is no period-1 solution $\{S_n\}$ to the N-DHM (34) which corresponds to any nontrivial solution $\{q_n\}$ to the AL-DNLS equation (18) under the action of discrete gauge transformations. In fact, if there is such a period-1 solution $\{S_n\}$, then the corresponding gauge matrix function $\{G_n\}$ should satisfy $G_{n+1}^{-1}\sigma_3 G_{n+1} = G_n^{-1}\sigma_3 G_n$ because of $S_{n+1} = S_n$, which implies $G_{n+1}G_n^{-1}$ commutes with σ_3. But we know from the first equation of (19) that $G_{n+1}G_n^{-1} = \begin{pmatrix} 1 & \bar{q}_n \\ -q_n & 1 \end{pmatrix}$, which does not commute with σ_3. This is a contradiction.

Now for the period-1 orbit solution (61), its yielding solution $\left\{q_n(t) = \sqrt{\frac{x_0}{2}}\,\mathrm{e}^{\mathrm{i}[(E-2)t-\theta_0]}\right\}$ to the AL-DNLS equation (18) is period 1 too. As just indicated above, there is no period-1 solution $\{S_n\}$ to the N-DHM (34), which corresponds this period-1 solution via a discrete gauge transformation. Another period-1 orbit solution (62) yields actually a period-2 solution to the AL-DNLS equation (18). This solution is explicitly written as

$$\begin{cases} q_0(t) = \sqrt{\dfrac{-x_0}{2}}\exp \mathrm{i}[(E-2)t-\theta_0], \\ q_1(t) = -\sqrt{\dfrac{-x_0}{2}}\exp \mathrm{i}[(E-2)t-\theta_0], \\ q_{n+2}(t) = q_n(t) \quad \forall n, \end{cases} \tag{64}$$

with $x_0 = -2\frac{E+2}{2+\gamma} < 0$, where θ_0 is the argument of φ_0. Its gauge corresponding solution $\{S_n\}$ is thus given by formula (41). Because of $q_n + q_{n+1} = 0$, one checks in this case (by using formulae (39), (40) with $n = 2$) that $G_{n+2} = \begin{pmatrix} 1+\frac{-x_0}{2} & 0 \\ 0 & 1+\frac{-x_0}{2} \end{pmatrix} G_n$ for all n. Therefore, the solution $\{S_n\}$ satisfies

$$S_{n+2} = G_{n+2}^{-1}\sigma_3 G_{n+2} = G_n^{-1}\sigma_3 G_n = S_n, \qquad \forall n. \tag{65}$$

This indicates that the solution $\{S_n\}$ is still of period 2. The explicit expressions of the solution $\{S_n\}$ can be deduced as follows. Substituting q_0 and q_1 given by (64) into the ODE (37) (for the purpose of simplicity, we take $\theta_0 = 0$), we solve it to get

$$G_0 = \begin{pmatrix} \left(\cos Ft + \mathrm{i}\frac{-x_0+E-2}{2F}\sin Ft\right)\mathrm{e}^{-\mathrm{i}\frac{E-2}{2}t} & -2\mathrm{i}\frac{\sqrt{\frac{-x_0}{2}}}{F}\mathrm{e}^{-\mathrm{i}\frac{E-2}{2}t}\sin Ft \\ -2\mathrm{i}\frac{\sqrt{\frac{-x_0}{2}}}{F}\mathrm{e}^{\mathrm{i}\frac{E-2}{2}t}\sin Ft & \left(\cos Ft - \mathrm{i}\frac{-x_0+E-2}{2F}\sin Ft\right)\mathrm{e}^{\mathrm{i}\frac{E-2}{2}t} \end{pmatrix},$$

where $F = \sqrt{\left(\frac{-x_0+E-2}{2}\right)^2 - 2x_0}$. Thus, the period-2 solution $\{S_n\}$ is explicitly expressed (we only need to give first two of it) as follows:

$$S_0 = G_0^{-1}\sigma_3 G_0$$

$$= \begin{pmatrix} \cos^2 Ft + \left[\left(\frac{-x_0+E-2}{2F}\right)^2 - \frac{-2x_0}{F^2}\right]\sin^2 Ft & -4\mathrm{i}\left(\cos Ft - \mathrm{i}\frac{\frac{-x_0+E-2}{2}}{F}\sin Ft\right)\frac{\sqrt{\frac{-x_0}{2}}}{F}\sin Ft \\ 4\mathrm{i}\left(\cos Ft + \mathrm{i}\frac{\frac{-x_0+E-2}{2}}{F}\sin Ft\right)\frac{\sqrt{\frac{-x_0}{2}}}{F}\sin Ft & -\cos^2 Ft - \left[\left(\frac{-x_0+E-2}{2F}\right)^2 - \frac{-2x_0}{F^2}\right]\sin^2 Ft \end{pmatrix},$$

$$S_1 = G_0^{-1}\frac{1}{1+\frac{-x_0}{2}}\begin{pmatrix} 1+\frac{x_0}{2} & 2\sqrt{\frac{-x_0}{2}}\,\mathrm{e}^{-\mathrm{i}(E-2)t} \\ 2\sqrt{\frac{-x_0}{2}}\,\mathrm{e}^{\mathrm{i}(E-2)t} & -1-\frac{x_0}{2} \end{pmatrix} G_0.$$

The period-doubling bifurcation solution (63) yields a period-4 solution to the AL-DNLS equation (18), which is expressed explicitly as

$$\begin{cases} q_0(t) = q_3(t) = \sqrt{\frac{x_0}{2}}\exp \mathrm{i}[(E-2)t - \theta_0], \\ q_1(t) = q_2(t) = \sqrt{\frac{x_0}{2}}\exp \mathrm{i}[(E-2)t - \pi - \theta_0], \\ q_{n+4}(t) = q_n(t), \quad \forall n, \end{cases} \tag{66}$$

where $x_0 = 2\left|\frac{E}{\gamma}\right|$. In this case, we see that the gauge corresponding solution $\{S_n\}$ to the N-DHM (34) satisfies $S_{n+4} = S_n$ $(\forall n)$. This is because of a direct calculation: $G_{n+4} = \begin{pmatrix} (1+\frac{x_0}{2})^2 & 0 \\ 0 & 1+(\frac{x_0}{2})^2 \end{pmatrix} G_n$ for all n. Thus, $\{S_n\}$ is still of period 4. In other words, the period-4 solution (66) of the AL-DNLS equation (18) is preserved under the action of the discrete gauge transformation.

Finally, we check if the period-1, period-2 orbit solutions (59), (60) in the case of $J \neq 0$ are preserved under the action of discrete gauge transformations. In the case of $\widetilde{\gamma} \geqslant 0$, let $x = \widetilde{x}_0$ be a real root of the algebraic equation

$$x = \frac{E + \widetilde{\gamma}\sqrt{x^2+1}}{2(1+\widetilde{\mu}\sqrt{x^2+1})}\sqrt{x^2+1}.$$

Then,

$$q_n(t) = \varphi_n \exp[\mathrm{i}(E-2)t] = r\exp \mathrm{i}[(E-2)t - n\alpha_0 - \theta_0] \tag{67}$$

is its yielding solution to the AL-DNLS equation (18), where θ_0 is the argument of φ_0 (without loss of generality, θ_0 is chosen to be zero), r associated with $J \neq 0$ satisfies the compatibility condition $\frac{\widetilde{x}_0^2}{16J^2} + J^2 = r^4$ from the scaling transformation and (46), and $\alpha_0 = \arcsin \frac{J}{r^2}$. This solution is not period in general. But it may be a period-m solution for a positive integer m if it happens $\alpha_0 = \frac{2\pi}{m}$. In this situation, we check that, for $m = 3$ or 4, gauge corresponding solution of (67) to (34) is not of the period m. It is also verified that, for the period-2 solution (60) which yields the following period-4 solution

$$q_n(t) = r \exp \mathrm{i}\left[(E-2)t - \left[\frac{n+1}{2}\right]\pi + \frac{1+(-1)^{n+1}}{2}\alpha_0\right], \quad \forall n \tag{68}$$

to the AL-DNLS equation (18) (where r and α_0 are the same ones appeared in (67)), the gauge corresponding solution $\{S_n\}$ to (34) is not of period 4.

Thus, roughly speaking, we have displayed the following interesting phenomenon. The nontrivial period solutions (i.e., period $\geqslant 2$) with the probability current $J = 0$ to the AL-DNLS equation (18) are transformed by discrete gauge transformation to solutions with the same period to the nonintegrable discrete Heisenberg model (34). However, the period solutions with the probability current $J \neq 0$ to the AL-DNLS equation (18) are transformed by discrete gauge transformation to solutions without the same period to the nonintegrable discrete Heisenberg model (34). This illustrates that the bifurcation-creating properties of the AL-DNLS equation (18) are conditionally preserved under the action of discrete gauge transformations. We should point out that the bifurcation properties of the two gauged discrete equations that are not exactly the same need not imply that the two dynamical systems are different. Even in the continuum case, for example, in polar coordinates of the Heisenberg model and the nonlinear Schrödinger equations they do not correspond to each other in all their properties. In order to get an one-to-one correspondence, [17, 18] modify the polar coordinates appropriately by introducing a linear time-dependent term. Therefore, it may be possible that all the bifurcation properties of the recurrence equation of the stationary discrete nonintegrable nonlinear Schrödinger equation are produced by another stationary discrete nonintegrable Heisenberg model by suitable stationary modification.

4. Conclusion

In this paper, we presented a geometric investigation of the equation introduced by Cai, Bishop and Gronbech-Jensen [7] that interpolates between integrable AL and nonintegrable DNLS equation. By using the terminology of discrete connection and associated discrete curvature, we show that the AL-DNLS equation (18) is discrete gauge equivalent to the integrable–nonintegrable discrete Heisenberg model (34) (theorem 1) and vice versa (lemma 2). As solitonic properties of the integrable AL equation are preserved to those of the integrable discrete Heisenberg spin model under the action of discrete gauge transformations, it is interesting and important to see whether chaotic properties of the AL-DNLS equation are preserved under the action of discrete gauge transformations.

The transmission property of the AL-DNLS equation is one of important features describing its chaotic dynamics. This property is proved to be preserved and delivered to that of the nonintegrable discrete Heisenberg model (34) under the action of discrete gauge transformations. It indicates that the AL-DNLS equation (18) and the nonintegrable discrete Heisenberg model (34) share completely the same transmitting behaviours (see figures 1–3). The bifurcation-creating property, another important dynamical feature, of the AL-DNLS equation (18) is proved to be conditionally preserved under the action of discrete gauge transformations. It depends on whether its probability current is zero or not. This shows that some quasi-period orbits of the AL-DNLS equation are destroyed by discrete gauge transformations. However, as pointed out at the end of the last section, this needs not imply that the two dynamical systems are different. Any way, the geometric study for the AL-DNLS equation in this paper reveals that some chaotic properties of different nonlinear discrete equations may happen to be related and interacted to each other, and this aspect deserves further investigation.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (10531090) and STCSM. The author sincerely thanks the referees for their nice comments and suggestions. He also thanks Dr Wei Lin for helping to produce the figures in the paper by Mathlab and valuable discussions.

Appendix

This appendix gives explicit proofs of (24), (25), (26), (30), (31) and the last step of (27).

(1) *The proof of (24).* By the first equation of (19) and the relation (22), we have

$$\begin{aligned} S_{n-1}S_n &= G_{n-1}^{-1}\sigma_3 G_{n-1}G_n^{-1}\sigma_3 G_n = G_n^{-1}G_nG_{n-1}^{-1}\sigma_3 G_{n-1}G_n^{-1}\sigma_3 G_n \\ &= G_n^{-1}\begin{pmatrix} 1 & \bar{q}_{n-1} \\ q_{n-1} & 1 \end{pmatrix}\sigma_3\begin{pmatrix} 1 & \bar{q}_{n-1} \\ q_{n-1} & 1 \end{pmatrix}^{-1}\sigma_3 G_n \\ &= \frac{1}{1+|q_{n-1}|^2}G_n^{-1}\begin{pmatrix} 1-|q_{n-1}|^2 & 2\bar{q}_{n-1} \\ -2q_{n-1} & 1-|q_{n-1}|^2 \end{pmatrix}G_n. \end{aligned} \tag{A.1}$$

Hence, $\frac{1}{2}\,\mathrm{tr}(S_{n-1}S_n) = \frac{1-|q_{n-1}|^2}{1+|q_{n-1}|^2}$ and $1+\frac{1}{2}\,\mathrm{tr}(S_{n-1}S_n) = \frac{2}{1+|q_{n-1}|^2}$. This proves (24).

(2) *The proof of (25).* From above (A.1), we see

$$\begin{aligned} I + S_{n-1}S_n &= G_n^{-1}\left[I + \begin{pmatrix} \frac{1-|q_{n-1}|^2}{1+|q_{n-1}|^2} & \frac{2\bar{q}_{n-1}}{1+|q_{n-1}|^2} \\ -\frac{2q_{n-1}}{1+|q_{n-1}|^2} & \frac{1-|q_{n-1}|^2}{1+|q_{n-1}|^2} \end{pmatrix}\right]G_n \\ &= \frac{2}{1+|q_{n-1}|^2}G_n^{-1}\begin{pmatrix} 1 & \bar{q}_{n-1} \\ -q_{n-1} & 1 \end{pmatrix}G_n. \end{aligned}$$

Hence,

$$G_n^{-1}\begin{pmatrix} 1 & \bar{q}_{n-1} \\ -q_{n-1} & 1 \end{pmatrix}G_n = \frac{1+|q_{n-1}|^2}{2}(I+S_{n-1}S_n) = \frac{I+S_{n-1}S_n}{1+\frac{1}{2}\,\mathrm{tr}(S_{n-1}S_n)}.$$

Here we have used equation (24). This completes the proof of (25).

(3) *The proof of (26).* Similar to getting (A.1), from the first equation of (19) and the relation (22), we have

$$\begin{aligned} S_n + S_{n-1} &= G_n^{-1}\sigma_3 G_n + G_{n-1}^{-1}\sigma_3 G_{n-1} = G_n^{-1}\left(\sigma_3 + G_nG_{n-1}^{-1}\sigma_3 G_{n-1}G_n^{-1}\right)G_n \\ &= G_n^{-1}\left[\sigma_3 + \begin{pmatrix} 1 & \bar{q}_{n-1} \\ q_{n-1} & 1 \end{pmatrix}\sigma_3\begin{pmatrix} 1 & \bar{q}_{n-1} \\ q_{n-1} & 1 \end{pmatrix}^{-1}\right]G_n \\ &= G_n^{-1}\left[\sigma_3 + \frac{1}{1+|q_{n-1}|^2}\begin{pmatrix} 1-|q_{n-1}|^2 & -2\bar{q}_{n-1} \\ -2q_{n-1} & -1+|q_{n-1}|^2 \end{pmatrix}\right]G_n \\ &= \frac{2}{1+|q_{n-1}|^2}G_n^{-1}\begin{pmatrix} 1 & -\bar{q}_{n-1} \\ -q_{n-1} & -1 \end{pmatrix}G_n. \end{aligned}$$

Hence,

$$G_n^{-1}\begin{pmatrix}1 & -\bar{q}_{n-1}\\ -q_{n-1} & -1\end{pmatrix}G_n = \frac{1+|q_{n-1}|^2}{2}(S_n + S_{n-1}) = \frac{S_n + S_{n-1}}{1+\frac{1}{2}\,\mathrm{tr}(S_{n-1}S_n)}.$$

Here we have used equation (24) too. This completes the proof of (26).

(4) *The proof of (30)*. Completely analogous to the above proof of (24), we have, from (19) and (22),

$$\begin{aligned}S_{n+1}S_n &= G_{n+1}^{-1}\sigma_3 G_{n+1}G_n^{-1}\sigma_3 G_n = G_n^{-1}G_nG_{n+1}^{-1}\sigma_3 G_{n+1}G_n^{-1}\sigma_3 G_n\\ &= \frac{1}{1+|q_{n-1}|^2}G_n^{-1}\begin{pmatrix}1-|q_n|^2 & -2\bar{q}_n\\ 2q_n & 1-|q_n|^2\end{pmatrix}G_n.\end{aligned} \tag{A.2}$$

Hence,

$$1-\frac{1}{2}\,\mathrm{tr}(S_{n+1}S_n) = 1-\frac{1-|q_n|^2}{1+|q_n|^2} = \frac{2|q_n|^2}{1+|q_n|^2},$$

$$1+\frac{1}{2}\,\mathrm{tr}(S_{n+1}S_n) = 1+\frac{1-|q_n|^2}{1+|q_n|^2} = \frac{2}{1+|q_n|^2}.$$

This shows (30).

(5) *The proof of (31)*. Like the above proof of (26), we have

$$\begin{aligned}S_{n+1}+S_n &= G_{n+1}^{-1}\sigma_3G_{n+1}+G_n^{-1}\sigma_3G_n = G_n^{-1}\big(G_nG_{n+1}^{-1}\sigma_3G_{n+1}G_n^{-1}+\sigma_3\big)G_n\\ &= G_n^{-1}\left[\begin{pmatrix}1 & \bar{q}_n\\ q_n & 1\end{pmatrix}^{-1}\sigma_3\begin{pmatrix}1 & \bar{q}_n\\ q_n & 1\end{pmatrix}+\sigma_3\right]G_n\\ &= \frac{2}{1+|q_n|^2}G_n^{-1}\begin{pmatrix}1 & \bar{q}_n\\ q_{n-1} & -1\end{pmatrix}G_n.\end{aligned}$$

Hence,

$$G_n^{-1}\begin{pmatrix}1 & \bar{q}_{n-1}\\ q_{n-1} & -1\end{pmatrix}G_n = \frac{1+|q_n|^2}{2}(S_{n+1}+S_n) = \frac{S_{n+1}+S_n}{1+\frac{1}{2}\,\mathrm{tr}(S_{n+1}S_n)}.$$

Here we have used equation (30). This completes the proof of (31).

Finally, we come to show the last step of (27), which was omitted in [12]. However, the proof is somewhat complicated.

(6) *The proof of the last step of (27)*. From (22), we see that $S_n^2 = I$. Similar to (A.2), we may also have

$$S_nS_{n+1} = \frac{1}{1+|q_{n-1}|^2}G_n^{-1}\begin{pmatrix}1-|q_n|^2 & 2\bar{q}_n\\ -2q_n & 1-|q_n|^2\end{pmatrix}G_n.$$

Combining this equation with (A.2), we obtain the following relation:

$$S_nS_{n+1}+S_{n+1}S_n = 2\frac{1-|q_n|^2}{1+|q_n|^2}I = -2I+\frac{4}{1+|q_n|^2}I. \tag{A.3}$$

By noting $\big(1-\frac{z^2+z^{-2}}{2}\big)\frac{z+z^{-1}}{2} = \frac{1}{4}(-z^3+z+z^{-1}-z^{-3})$, $\frac{z^2-z^{-1}}{2}\frac{z+z^{-1}}{2} = \frac{1}{4}(z^3+z-z^{-1}-z^{-3})$, $\big(1-\frac{z^2+z^{-2}}{2}\big)\frac{z-z^{-1}}{2} = \frac{1}{4}(-z^3+3z-3z^{-1}+z^{-3})$ and $\frac{z^2-z^{-1}}{2}\frac{z-z^{-1}}{2} = \frac{1}{4}(z^3-z-z^{-1}+z^{-3})$, we see that

$$
\begin{aligned}
&-\mathrm{i}\left(1-\frac{z^2+z^{-2}}{2}\right)\frac{z+z^{-1}}{2}\frac{S_{n+1}+S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}+\mathrm{i}\frac{z^2-z^{-2}}{2}\frac{z+z^{-1}}{2}\frac{I+S_nS_{n+1}}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}\\
&\quad-\mathrm{i}\left(1-\frac{z^2+z^{-2}}{2}\right)\frac{z-z^{-1}}{2}\frac{(S_{n+1}+S_n)S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}+\mathrm{i}\frac{z^2-z^{-2}}{2}\frac{z-z^{-1}}{2}\frac{(I+S_nS_{n+1})S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}\\
&\quad=-\frac{\mathrm{i}}{4}(-z^3+z+z^{-1}-z^{-3})\frac{S_{n+1}+S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}+\frac{\mathrm{i}}{4}(z^3+z-z^{-1}-z^{-3})\frac{I+S_nS_{n+1}}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}\\
&\quad-\frac{\mathrm{i}}{4}(-z^3+3z-3z^{-1}+z^{-3})\frac{S_{n+1}S_n+I}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}\\
&\quad+\frac{\mathrm{i}}{4}(z^3-z-z^{-1}+z^{-3})\frac{-S_{n+1}-S_n+\frac{4}{1+|q_n|^2}S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}\\
&\quad=\frac{\mathrm{i}}{4}(z^3+z-z^{-1}-z^{-3})\frac{S_nS_{n+1}}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}+\frac{\mathrm{i}}{4}(z^3-3z+3z^{-1}-z^{-3})\frac{S_{n+1}S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}\\
&\quad+\frac{\mathrm{i}}{2}(z^3-z+z^{-1}-z^3)\frac{I}{1+\operatorname{tr}(S_{n+1}S_n)}+\frac{\mathrm{i}}{4}(z^3-z-z^{-1}+z^{-3})\frac{\frac{4}{1+|q_n|^2}S_n}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}.
\end{aligned}
\tag{A.4}
$$

Now, by (A.3), substituting $S_{n+1}S_n=-S_nS_{n+1}-2I+\frac{4}{1+|q_n|^2}I$ into the second term on the right-hand side of (A.4), we see that the coefficient of $\frac{S_nS_{n+1}}{1+\frac{1}{2}\operatorname{tr}(S_{n+1}S_n)}$ is $\mathrm{i}(z-z^{-1})$. Then, substituting $S_nS_{n+1}=\frac{1}{2}\big(S_nS_{n+1}-S_{n+1}S_n-2I+\frac{4}{1+|q_n|^2}I\big)$ again into the resulted expression, we arrive finally at

$$
\begin{aligned}
\text{right-hand side of (A.4)}&=-\frac{\mathrm{i}}{2}(z-z^{-1})\frac{S_{n+1}S_n-S_nS_{n+1}}{1+\operatorname{tr}(S_{n+1}S_n)}+\frac{\mathrm{i}}{2}(z^3-z+z^{-1}-z^{-3})I\\
&\quad+\frac{\mathrm{i}}{2}(z^3-z-z^{-1}+z^{-3})S_n.
\end{aligned}
\tag{A.5}
$$

In the same computation displayed in getting (A.5) and by using the relation

$$
S_nS_{n-1}+S_{n-1}S_n=2\frac{1-|q_{n-1}|^2}{1+|q_{n-1}|^2}I=-2I+\frac{4}{1+|q_{n-1}|^2}I,
$$

we also have

$$
\begin{aligned}
&\mathrm{i}\left(1-\frac{z^2+z^{-2}}{2}\right)\frac{z+z^{-1}}{2}\frac{S_n+S_{n-1}}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})}-\mathrm{i}\frac{z^2-z^{-2}}{2}\frac{z+z^{-1}}{2}\frac{I+S_{n-1}S_n}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})}\\
&\quad+\mathrm{i}\left(1-\frac{z^2+z^{-2}}{2}\right)\frac{z-z^{-1}}{2}\frac{S_n(S_n+S_{n-1})}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})}-\mathrm{i}\frac{z^2-z^{-2}}{2}\frac{z-z^{-1}}{2}\frac{S_n(I+S_{n-1}S_n)}{1+\frac{1}{2}\operatorname{tr}(S_nS_{n-1})}\\
&=\frac{\mathrm{i}}{2}(z-z^{-1})\frac{S_nS_{n-1}-S_{n-1}S_n}{1+\operatorname{tr}(S_nS_{n-1})}-\frac{\mathrm{i}}{2}(z^3-z+z^{-1}-z^{-3})I-\frac{\mathrm{i}}{2}(z^3-z-z^{-1}+z^{-3})S_n.
\end{aligned}
\tag{A.6}
$$

Hence, we obtain the last (i.e., third) equality in (27) by summing (A.5) and (A.6) and substituting the resulted summation into the expression of the second equality in (27). This finishes the proof of the last step of (27).

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